ON THE DEFINITION
OF REDUCIBLE HYPERCOMPLEX NUMBER SYSTEMS*

BY

SAUL EPSTEEN

According to PEIRCE† and SCHEFFERS‡ a hypercomplex number system is said to be reducible when, by a suitable choice of units

$$E \equiv E_j E_k \equiv e_1 \cdots e_m e_{m+1} \cdots e_n,$$

with the relations

$$e_{i_1} e_{i_2} = \sum_{i_3} \gamma_{i_1 i_2 i_3} e_{i_3},$$

the following conditions are fulfilled: §

C_1) E_j forms a system by itself, i. e.,

$$e_{j_1} e_{j_2} = \sum_{j_3} \gamma_{j_1 j_2 j_3} e_{j_3} \qquad (\gamma_{j_1 j_2 k} = 0);$$

C_2) E_k forms a system by itself, i. e.,

$$e_{k_1} e_{k_2} = \sum_{k_3} \gamma_{k_1 k_2 k_3} e_{k_3} \qquad (\gamma_{k_1 k_2 j} = 0);$$

A) $$e_j e_k = 0 \qquad (\gamma_{jki} = 0);$$

B) $$e_k e_j = 0 \qquad (\gamma_{kji} = 0).$$

One of the chief results of this paper is that *for systems E containing a modulus the conditions C_1, C_2 are unnecessary, being consequences of the others.* In other words, *a hypercomplex number system containing a modulus is reducible if:*

$$A)\ e_j e_k = 0 \qquad \textit{and} \qquad B)\ e_k e_j = 0.$$

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† BENJ. PEIRCE, American Journal of Mathematics, vol. 4 (1881), p. 100.

‡ SCHEFFERS, Mathematische Annalen, vol. 39 (1891), p. 317.

§ In this paper the scheme of subscripts is the following:

$$i = 1, \cdots, n;\ j = 1, \cdots, m;\ k = m + 1, \cdots, n.$$

1. *Preliminary Remarks.*

It is assumed that the hypercomplex number system $E \equiv e_1 \cdots e_n$ is associative, that is, that

C_a) $$(e_{i_1} e_{i_2}) e_{i_3} = e_{i_1} (e_{i_2} e_{i_3}).$$

Letting

$$X = \sum_{i_1} x_{i_1} e_{i_1}, \qquad Y = \sum_{i_2} y_{i_2} e_{i_2}$$

be two general numbers of the system, then we have

$$X' = XY = \sum_{i_3} x'_{i_3} e_{i_3} = \sum_{i_1 i_2} x_{i_1} y_{i_2} e_{i_1} e_{i_2}; \qquad x'_{i_3} = \sum_{i_1 i_2} \gamma_{i_1 i_2 i_3} x_{i_1} y_{i_2}.$$

The necessary and sufficient condition that the equation for Y,

$$0 = XY$$

X being given), have a solution $Y \neq 0$, in addition to $Y = 0$, is that

$$\Delta_x \equiv \left| \sum_{i_1} \gamma_{i_1 i_2 i_3} x_{i_1} \right| = 0 \qquad (i_2, i_3 = 1, \cdots, n).$$

In this case X is said to be a left hand divisor of zero. Further $\Delta_x \neq 0$ is the necessary and sufficient condition that in $X' = XY$ the left hand quotient Y of any X' by X be existent (in fact uniquely).

Hence the fulfillment of the condition

C_l) $$\Delta_x \equiv \left| \sum_{i_1} \gamma_{i_1 i_2 i_3} x_{i_1} \right| \not\equiv 0 \qquad (i_2, i_3 = 1, \cdots, n)$$

implies that

C_l) not every X is a left hand divisor of zero.

Similarly, the fulfilment of the condition

C_r) $$\Delta'_y \equiv \left| \sum_{i_2} \gamma_{i_1 i_2 i_3} y_{i_2} \right| \not\equiv 0 \qquad (i_1, i_3 = 1, \cdots, n).$$

implies that

C_r) not every Y is a right hand divisor of zero.

From C_a, C_l, C_r it follows that

C_m) the hypercomplex number system contains a modulus — that is to say, a number $\epsilon = \sum_i \epsilon_i e_i$ exists such that for every x

$$\epsilon x = x \epsilon = x;$$

and conversely, from C_a, C_m follow C_l and C_r.

We propose to investigate the system of seven conditions:

$$C_a, \; C_l, \; C_r, \; C_1, \; C_2, \; A, \; B,$$

and to prove that in this system the conditions C_1, C_2 are redundant and the remaining five conditions are mutually independent.

2. *Redundancy of C_1 and C_2.*

I. We regard as given the hypercomplex number system $E \equiv e_1 \cdots e_n$. The units are parted into two sets, E_j and E_k,

$$E_j \equiv e_1 \cdots e_m: \qquad E_k = e_{m+1} \cdots e_n$$

and the general number of E, namely, $X = x_1 e_1 + \cdots + x_n e_n$, may be considered as the sum of two parts or components

$$X = J + K \qquad \left(J = \sum_j x_j e_j,\ K = \sum_k x_k e_k\right).$$

In this notation the condition C_1 states that $J_1 J_2 = J_3$,* viz., the product of any two J numbers is a J number. We proceed to prove that C_1 is a consequence of C_a, A, B, C_l.†

Setting

(1) $$J_1 J_2 = J_3 + K_3$$

we prove that

(2) $$XK_3 = 0$$

for every X, and hence, as desired, that $K_3 = 0$, since by C_l not every X is a left hand divisor of 0. We set

(3) $$X = J + K.$$

Then (2) follows at once from the two relations

(4) $$JK_3 = 0, \qquad KK_3 = 0,$$

of which the former holds by A, and the latter is evident from (1) by left hand multiplication by K and the use of C_a and B.

Hence C_1 is a consequence of C_a, C_l, A, B.

Similarly, C_1 is a consequence of C_a, C_r, A, B. For from C_a, A, B it follows that $K_3 X = 0$ for every X.

By parity of reasoning, C_2 is a consequence of C_a, A, B and C_l or C_r.

* $J_1 = \sum_j x_{1j} e_j$, $J_2 = \sum_j x_{2j} e_j$, etc.

† By working explicitly with the units the theorem first demonstrated was that C_1 is a consequence of C_a, A, B, C_2, C_l. The suggestion of Professor E. H. Moore to work immediately with the numbers and their components led me to the discovery that C_1 *can be demonstrated without assuming* C_2. The method employed for proving the independence of C_l and C_r (§ 5) is also due to Professor Moore.

Hence, *C_1 and C_2 are consequences of C_a, A, B, C_l or C_r, and a fortiori of C_a, A, B, C_m*, that is, *in an associative hypercomplex number system containing a modulus, C_1 and C_2 are consequences of A and B.*

We proceed to prove the mutual independence* of C_a, C_l, C_r, A, B.

3. *Independence of A and B.*

Consider the irreducible system † whose multiplication table is

$$E: \quad \begin{array}{c|ccc} & e_1 & e_2 & e_3 \\ \hline e_1 & 0 & 0 & e_1 \\ e_2 & e_1 & e_2 & 0 \\ e_3 & 0 & 0 & e_3 \end{array}.$$

Here we have $E \equiv E_j E_k \equiv (e_1 e_2)(e_3)$. The condition C_a is fulfilled, and, as E contains a modulus, namely, $\epsilon = e_2 + e_3$, the conditions C_l and C_r are both fulfilled.

Since $e_3 e_1 = e_3 e_2 = 0$ the condition B (viz.: $e_k e_j = 0$) is also fulfilled. But $e_j e_k$ does not vanish for every j and k, for example $e_1 e_3 = e_1 (\neq 0)$. Thus C_a, C_l, C_r, B are satisfied and A is contradicted. Hence A is independent of C_a, C_l, C_r. B.

By interchanging j and k one sees that B is independent of C_a, C_l, C_r, A.

4. *Independence of C_a.*

Consider the set of units e_1, e_2 whose multiplication table is

$$\begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & e_2 & 0 \\ e_2 & 0 & e_1 \end{array}.$$

The determinants Δ_x and Δ'_y are

$$\Delta_x \equiv \begin{vmatrix} 0 & x_2 \\ x_1 & 0 \end{vmatrix}, \qquad \Delta'_y \equiv \begin{vmatrix} 0 & y_2 \\ y_1 & 0 \end{vmatrix}$$

and do not vanish identically; therefore the conditions C_l and C_r are fulfilled.

Since $e_1 e_2 = e_2 e_1 = 0$ the conditions A and B are also fulfilled. But the condition C_a is contradicted, since

* That C_a, C_l, C_r are independent was proved by Professor L. E. DICKSON in these Transactions, vol. 4 (1903), pp. 21–24.

† SCHEFFERS, l. c., p. 343.

$$(e_1 e_1) e_2 = e_2 \cdot e_2 = e_1 \neq e_1(e_1 e_2) = e_1 \cdot 0 = 0 .$$

Hence C_a is independent of C_r, C_l, A, B.

5. *Independence of C_l and C_r.*

I. Consider the two systems

$$E'_j: \quad \begin{array}{c|cc} & e_1 & e_2 \\ \hline e_1 & 0 & e_1 \\ e_2 & 0 & e_2 \end{array}, \qquad E'_k \cdot \quad \begin{array}{c|cc} & e_3 & e_4 \\ \hline e_3 & 0 & e_3 \\ e_4 & 0 & e_4 \end{array} .$$

It is easily verified that in each system C_a is fulfilled. Therefore the system $E' = E'_j \, E'_k$:

$$E': \quad \begin{array}{c|cccc} & e_1 & e_2 & e_3 & e_4 \\ \hline e_1 & 0 & e_1 & 0 & 0 \\ e_2 & 0 & e_2 & 0 & 0 \\ e_3 & 0 & 0 & 0 & e_3 \\ e_4 & 0 & 0 & 0 & e_4 \end{array},$$

also satisfies the condition C_a. The conditions A and B are satisfied in E'.

For this system

$$\Delta_x \equiv \begin{vmatrix} 0 & x_1 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & x_3 \\ 0 & 0 & 0 & x_4 \end{vmatrix} \equiv 0, \qquad \Delta'_y \equiv \begin{vmatrix} y_2 & 0 & 0 & 0 \\ 0 & y_2 & 0 & 0 \\ 0 & 0 & y_4 & 0 \\ 0 & 0 & 0 & y_4 \end{vmatrix} \not\equiv 0;$$

thus the condition C_r is fulfilled and C_l is not fulfilled.

Therefore C_l is independent of C_a, C_r, A, B.

II. By interchanging *right* and *left* we see at once that C_r is independent of C_a, C_l, A, B.

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